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Extrema of Landau polynomials†

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Abstract. For Landau polynomials which are invariant under some representation of a finite group and which satisfy a non-degeneracy condition two methods of calculating the number of orbits of complex critical points with a given isotropy subgroup are given. Neither method uses the explicit form of the polynomial.

1. Introduction

This note describes two methods of calculating the number of orbits of complex critical points with a given isotropy subgroup, for Landau polynomials which are invariant under some representation of a finite group and which satisfy certain non-degeneracy conditions. The methods differ in the information they need about the group and in their range of applicability. Both methods require a knowledge of which subgroups of the group appear as isotropy subgroups for the representation concerned; in addition the first method, given in § 2, uses the character table of the group while the second, described in § 3, uses the 'table of marks' (Burnside 1897) of the set of (conjugacy classes of) isotropy subgroups. Since the character table of a group is frequently available, whereas the table of marks is relatively unknown, the first method is usually more convenient. However, it has the disadvantage of not always working (as the example in § 5.3 shows) whereas the second method is guaranteed to give an answer.

Let G be a finite group which acts on $\mathbb{R}^m$ by means of a faithful representation $D: G \to O(m)$ and let $f(x; \alpha)$ be a polynomial function of $x \in \mathbb{R}^m$ depending on a vector of parameters α and which is invariant under the action of G, i.e.

$$f(D(g) \cdot x; \alpha) = f(x; \alpha) \qquad \text{for all } g \in G.$$

Write $f(x; \alpha)$ as

$$f_\alpha(x) = f(x; \alpha) = f^{(d)}(x; \alpha) + f^{(d-1)}(x; \alpha) + \ldots + f^{(1)}(x; \alpha) + f^{(0)}(\alpha)$$

where $f^{(r)}(x; \alpha)$ is homogeneous of degree r in x. The non-degeneracy condition that f_α is required to satisfy is as follows.

(C) All the complex critical points of f_α are non-degenerate, i.e. if $x_0 \in \mathbb{C}^m$ is a critical point, the Hessian matrix $(\partial^2 f/\partial x_i\, \partial x_j)|_{x=x_0}$ has rank m. It follows that the critical points are isolated and finite in number.

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For example, the polynomial

$$f(x, y; \alpha) = a(x^4 + y^4) + bx^2y^2 + c(x^2 + y^2) + d$$

will satisfy (C) if and only if $a \neq 0$, $b \neq \pm 2a$ and $c \neq 0$.

For a given f, the set of parameter values α such that $f(\cdot\,; \alpha)$ satisfies (C) will either be empty or will be the complement of a subset of the parameter space given by the zeros of a set of polynomial equations and hence will be open and dense. For any finite group G and representation D a polynomial function f of the second type can always be found, if the degree d is allowed to be large enough; in practice $d = 4$ is usually sufficient though sometimes it is necessary to go to $d = 6$.

2. Method 1

The set of all isotropy subgroups of points $x \in \mathbb{R}^m$ is divided into conjugacy classes, the conjugacy class of $H < G$ being denoted by (H). Let $\{H_i\}_{i \in I}$ be a set of isotropy subgroups, one taken from each conjugacy class. If x is a critical point of an invariant function f, then so is $D(g) \cdot x$ for all $g \in G$; I shall call the orbit $\{D(g) \cdot x; g \in G\}$ a critical orbit of f. If x has isotropy subgroup H then $D(g) \cdot x$ has isotropy subgroup gHg^{-1}; so to each critical orbit of f there is associated a conjugacy class of isotropy subgroups of G.

Let f_α be an invariant Landau polynomial satisfying (C) and let Ω be the set of critical points of f_α. By Bezout's theorem Ω contains $(d-1)^m$ points and, by the remarks above, these points are permuted by the action of G; thus Ω is a 'G-set' of order $(d-1)^m$ (Serre 1977). The action of G on Ω induces a linear action on the vector space $\mathrm{Map}(\Omega, \mathbb{C})$, of all mappings of Ω into $\mathbb{C}$ giving a representation of G called the 'permutation representation' of Ω. The character χ of this representation is easily seen to be given by

$$\chi(g) = \text{the number of critical points of } f \text{ fixed by } g \in G.$$

For $g \in G$, f_α has a critical point at $x \in \mathrm{Fix}(g)$ if and only if the restriction of f_α to $\mathrm{Fix}(g)$ has a critical point at x (this is a special case of the 'principal of symmetric criticality' (Palais 1979); using this and the symmetry of the Hessian of f_α at a critical point, it can be shown that, for each g, the restriction of f_α to the fixed point set of g also satisfies (C) and so will have $(d-1)^{l(g)}$ complex critical points, where $l(g) = \dim \mathrm{Fix}(g)$; these will be precisely the critical points of f_α which lie in $\mathrm{Fix}(g)$. In other words:

Theorem 1.

$$\chi(g) = (d-1)^{l(g)}.$$

If f_α has $n_{(H_i)}$ critical orbits with isotropy groups in (H_i), then Ω is equal to the disjoint union

$$\Omega = \coprod_{i \in I} n_{(H_i)} \cdot (G/H_i)$$

(where $n_{(H_i)} \cdot (G/H_i)$ means the disjoint union of $n_{(H_i)}$ copies of G/H_i). The character of the permutation representation of a union of G-sets is the sum of the characters of the individual G-sets. Since the character of the permutation representation of G/H_i

is $1_{H_i}^G$, the character of G induced from the trivial character 1_{H_i} of H_i, it follows that

$$\chi = \sum_{i \in I} n_{(H_i)} 1_{H_i}^G.$$

Let $\{\chi_j\}_{j \in J}$ be a complete set of irreducible characters of G; then the above expression implies the following.

Theorem 2.

$$\langle \chi, \chi_j \rangle = \sum_{i \in I} n_{(H_i)} \langle 1_{H_i}^G, \chi_j \rangle, \qquad j \in J,$$

where $\langle\ ,\ \rangle$ is the usual 'inner product' of characters

$$\langle \chi', \chi'' \rangle = (1/G) \sum_{g \in G} \chi'(g) \chi''(g^{-1}).$$

Since both $\langle \chi, \chi_j \rangle$ and $\langle 1_{H_i}^G, \chi_j \rangle$ can be calculated from a character table of G (using theorem 1 in the first case and Frobenius reciprocity (Serre 1977) in the second), theorem 2 gives a set of linear equations for the integers $n_{(H_i)}$. Unfortunately these equations do not always have a unique solution, as the example in § 5.3 shows; however, such cases seem to be rare, and when they do occur the method of § 3 can be used.

Theorem 2 also leads to an explicit expression for the total number of complex critical orbits of f. For any i we have $\langle 1_{H_i}^G, 1_G \rangle = 1$ and so

$$\langle \chi, 1_G \rangle = \sum_{i \in I} n_{(H_i)} \langle 1_{H_i}^G, 1_G \rangle = \sum_{i \in I} n_{(H_i)}$$

and we have the following corollary.

Corollary 3. The total number of complex critical orbits of f is equal to

$$(1/G) \sum_{g \in G} (d-1)^{l(g)}.$$

3. Method 2

In the second method of calculating the integers $n_{(H_i)}$, $i \in I$, Bezout's theorem is applied to the restriction of f_α to $\text{Fix}(H_i)$ for each $i \in I$, instead of $\text{Fix}(g)$. Thus, for essentially the same reasons as in § 2, if $m(H_i) = \dim \text{Fix}(H_i)$ the restriction of f_α to $\text{Fix}(H_i)$ will have $(d-1)^{m(H_i)}$ complex critical points and these will be precisely the critical points of f_α which lie in $\text{Fix}(H_i)$.

For each pair of isotropy groups H_i, H_j let $e(H_i, H_j)$ be the number of fixed points of the action of H_i on G/H_j obtained by restricting the natural action of G; in (Burnside 1897) $e(H_i, H_j)$ is called the 'mark' of H_i on H_j. Then the number of critical points of f_α fixed by H_i is clearly equal to $\Sigma_{j \in J}\, e(H_i, H_j) n_{(H_j)}$ and so we have theorem 4.

Theorem 4.

$$\sum_{j \in J} e(H_i, H_j) n_{(H_j)} = (d-1)^{m(H_i)}.$$

This theorem gives a set of linear equations for the $n_{(H_i)}$, analogous to theorem 2. However, this set of equations always has a unique solution, since the matrix $(e(H_i, H_j))$

is invertible. To show this, first note that the set $\{(H_i)\}_{i\in I}$ of conjugacy classes of points $x\in\mathbb{R}^m$ can be partially ordered by letting $(H_i)<(H_j)$ if there exists $g\in G$ such that $gH_ig^{-1}\subset H_j$. If $(H_i)\not\leq(H_j)$ then $e(H_i, H_j)=0$ and, if $(H_i)=(H_j)$, $e(H_i, H_j)=|N(H_i)/H_i|\neq 0$ where $N(H_i)$ is the normaliser of H_i in G). It follows that I can be ordered so that $(e(H_i, H_j))$ is a lower triangular matrix with non-zero elements on the diagonal and so is invertible.

In fact the solution can be made explicit (using a generalisation of the Mobius inversion formula (Hall 1967)) as follows. Define $\bar{e}(H_i, H_j)$ inductively by

$$\bar{e}(H_i, H_i)=1/e(H_i, H_i)$$

and

$$\bar{e}(H_i, H_j)=\begin{cases}0 & (H_j)\not\geq(H_i),\\ -\dfrac{1}{e(H_i, H_j)}\displaystyle\sum_{k:(H_j)\geq(H_k)>(H_i)} e(H_i, H_k)\bar{e}(H_k, H_j), & (H_j)\geq(H_i).\end{cases}$$

Then $(\bar{e}(H_i, H_j))$ is the inverse of $(e(H_i, H_j))$, and so we have:

Corollary 5.

$$n_{(H_i)}=\sum_{j:(H_j)\geq(H_i)}\bar{e}(H_i, H_j)(d-1)^{m(H_j)}.$$

Adding together these expressions for the $n_{(H_i)}$ gives a result analogous to corollary 3.

Corollary 6. The total number of complex critical orbits of f_α is equal to

$$\sum_{i,j}\bar{e}(H_i, H_j)(d-1)^{m(H_j)}.$$

The advantage of this method over that of § 2 is clear; it always works. Its disadvantage is simply that the matrix of marks, $e(H_i, H_j)$, of a group is not as well known as the character table.

4.

An isotropy subgroup of G is said to be 'maximal' if it is maximal in the set of isotropy subgroups of points $x\in\mathbb{R}^m\setminus\{0\}$ with respect to the ordering given by inclusion. For a long time it was conjectured that, under suitable conditions, the absolute minima of Landau polynomials always have maximal isotropy subgroups (Michel 1983); however, a counterexample has recently been found (Jaric 1983). Here I shall give a simply checked sufficient condition for the conjecture to be true. This follows from the following theorem which is proved by applying Bezout's theorem to the restriction of f_α to Fix(H).

Theorem 7. If (H) is a conjugacy class of maximal isotropy subgroups and f_α satisfies condition (C) and has degree ≥ 3, then it has a complex critical orbit with isotropy groups in (H).

Let c be the number of conjugacy classes of maximal isotropy subgroups.

Corollary 8. If f_α is as in theorem 4, then it has at least $1+c$ complex critical orbits. If the number of complex critical orbits is equal to $1+c$, then they all (excluding that at the origin) have maxiamal isotropy subgroups.

This together with the formula of corollary 3 or corollary 5, calculating the number of complex critical orbits, gives the required sufficient condition.

5. Examples

5.1.

Let $G = O_h$ and D be the three-dimensional representation given by the symmetries of a cube. The group G has 48 elements, 15 leaving only the origin fixed, 23 fixing a one-dimensional subspace, 9 a two-dimensional subspace and only the identity fixing the whole of $\mathbb{R}^3$. If f_α is a Landau polynomial of degree 4 satisfying (C) (these exist and therefore, by a remark in § 1, occupy an open dense subset of the space of all invariant polynomial functions of degree 4) then it will have (by corollary 3)

$$\tfrac{1}{48}(15\times 1+23\times 3+9\times 9+1\times 27)=4$$

complex critical orbits.

The isotropy subgroups of the action of O_h on $\mathbb{R}^3$ are:

(i) G at the origin,

(ii) subgroups isomorphic to D_4, D_3 and D_2 along the three different types of axes of symmetry,

(iii) cyclic groups of order 2 on the planes of symmetry,

(iv) the trivial group elsewhere.

The conjugacy classes are the obvious ones, except that the cyclic groups of order 2 divide into two classes, corresponding to the two types of symmetry planes. They are denoted by

$$G, (D_4), (D_3), (D_2), (C_2), (C_2'), 1.$$

The maximal conjugacy classes are (D_4), (D_3) and (D_2) and a polynomial of degree 4 satisfying (C) can, by corollary 8, have only these and G as conjugacy classes of isotropy subgroups of critical orbits. A similar calculation for a degree-6 polynomial would show that f_α has ten complex critical orbits; using method 1 would show that these ten are divided up as:

1 orbit with isotropy subgroup G (the origin),

2 orbits with isotropy subgroups in (D_r) for each $r = 2, 3, 4$,

1 orbit with isotropy subgroups in (C_2),

2 orbits with isotropy subgroups in (C_2').

For an explicit calculation of the extrema in this example see (Jaric 1982).

5.2.

$G = D_6$ and D is the two-dimensional representation of D_6 given by the symmetries of a regular hexagon in the plane. To find an invariant polynomial satisfying (C) it is necessary to take d at least 6. The group D_6 has twelve elements, the identity, five non-trivial rotations which fix only the origin and six reflections which fix the lines of symmetry of the hexagon. Applying theorem 3 shows that any invariant polynomial, f_α, of degree 6 satisfying (C) must have

$$\tfrac{1}{12}(1\times 25+6\times 5+5\times 1)=5$$

complex critical orbits.

The conjugacy classes of isotropy subgroups are:

(i) G at the origin,

(ii) two conjugacy classes, (C_2), (C_2'), of cyclic groups of order 2 on the reflection planes,

(iii) 1 elsewhere.

The maximal conjugacy classes are (C_2) and (C_2').

Corollary 8 cannot be applied to deduce that the five critical orbits of f_α must have maximal isotropy subgroups so theorem 2 is used. It turns out that the one-dimensional characters are sufficient for the calculation; these are given by the following table.

	χ_1	χ_2	χ_3	χ_4
r^k	1	1	$(-1)^k$	$(-1)^k$
sr^k	1	-1	$(-1)^k$	$(-1)^{k+1}$

($k=0,\ldots,5$; the r^k are the rotations in D_6, the sr^k the reflections.)

χ is given by theorem 1 and a straightforward calculation shows that

$$\langle\chi,\chi_1\rangle=5,\qquad \langle\chi,\chi_2\rangle=0,$$
$$\langle\chi,\chi_3\rangle=2,\qquad \langle\chi,\chi_4\rangle=2.$$

Also

$$\langle 1_G^G,\chi_i\rangle=\begin{cases}1 & \text{if } i=1,\\ 0 & \text{otherwise,}\end{cases}$$

$$\langle 1_{C_2}^G,\chi_i\rangle=\begin{cases}1, & i=1,3,\\ 0, & i=2,4,\end{cases}$$

$$\langle 1_{C_2'}^G,\chi_i\rangle=\begin{cases}1, & i=1,4,\\ 0, & i=2,3,\end{cases}$$

$$\langle 1_1^G,\chi_i\rangle=1,\qquad i=1,2,3,4.$$

Substituting into the equations of theorem 2 gives

$$5=n_G+n_{(C_2)}+n_{(C_2')}+n_1,\qquad 0=n_1,$$
$$2=n_{(C_2)}+n_1,\qquad 2=n_{(C_2')}+n_1.$$

So $n_G=1$, $n_{(C_2)}=2$, $n_{(C_2')}=2$ and $n_1=0$ and, in fact, all critical orbits (except that of 0) have maximal isotropy subgroups.

5.3.

Let $G = D_2 = \{1, g, h, gh\}$ and D be the regular representation of G, that is the four-dimensional permutation representation associated to the G-set consisting of G acting on itself by multiplication.

The isotropy subgroups of D are G itself, the trivial group 1 and the cyclic groups $C_g = \langle 1, g\rangle$, $C_h = \langle 1, h\rangle$, $C_{gh} = \langle 1, gh\rangle$. The cyclic groups of order 1 are pairwise non-conjugate and each has a two-dimensional fixed point set; G has a one-dimensional fixed point set.

Method 1
The character table of G is:

	χ_1	χ_2	χ_3	χ_4
1	1	1	1	1
g	1	1	−1	−1
h	1	−1	1	−1
gh	1	−1	−1	1

Consider a non-degenerate Landau polynomial of degree 4; if χ is given by theorem 1, then

$$\chi(g) = \chi(h) = \chi(gh) = 9 \qquad \text{and} \qquad \chi(1) = 81$$

and

$$\langle \chi, \chi_1\rangle = 27, \qquad \langle \chi, \chi_i\rangle = 18, \qquad i = 2, 3, 4.$$

Also

$$\langle 1^G_{C_g}, \chi_i\rangle = \begin{cases} 1, & \text{if } i = 1, 2, \\ 0, & \text{if } i = 3, 4, \end{cases}$$

$$\langle 1^G_{C_h}, \chi_i\rangle = \begin{cases} 1, & \text{if } i = 1, 3, \\ 0, & \text{if } i = 2, 4, \end{cases}$$

$$\langle 1^G_{C_{gh}}, \chi_i\rangle = \begin{cases} 1, & \text{if } i = 1, 4, \\ 0, & \text{if } i = 2, 3. \end{cases}$$

Substituting into the equations of theorem 2 gives

$$27 = n_G + n_{C_g} + n_{C_h} + n_{C_{gh}} + n_1,$$

$$18 = n_{C_g} + n_1, \qquad 18 = n_{C_h} + n_1, \qquad 18 = n_{C_{gh}} + n_1.$$

The positive integer solutions are

$$n_1 = a, \qquad n_{C_g} = n_{C_h} = n_{C_{gh}} = 18 - a, \qquad n_G = 2a - 27,$$

$$a = 14, 15, 16, 17, 18,$$

so the solution is not unique.

Method 2

The table of marks of G is:

	1	C_g	C_h	C_{gh}	G
1	4	0	0	0	0
C_g	2	2	0	0	0
C_h	2	0	2	0	0
C_{gh}	2	0	0	2	0
G	1	1	1	1	1

Thus theorem 4 gives the equations

$$3 = n_G, \qquad 9 = 2n_{C_g} + n_G, \qquad 9 = 2n_{C_h} + n_G,$$
$$9 = 2n_{C_{gh}} + n_G, \qquad 81 = 4n_1 + 2n_{C_g} + 2n_{C_h} + 2n_{C_{gh}} + n_G,$$

which have the unique solution

$$n_G = 3 = n_{C_g} = n_{C_h} = n_{C_{gh}}, \qquad n_1 = 15.$$

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